BSVIEs with stochastic Lipschitz coefficients and applications in finance*

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Abstract

This paper is concerned with existence and uniqueness of M-solutions of backward stochastic Volterra integral equations (BSVIEs for short) which Lipschitz coefficients are allowed to be random, which generalize the results in [15]. Then a class of continuous time dynamic coherent risk measure is derived, allowing the riskless interest rate to be random, which is different from the case in [15].

Keywords: Backward stochastic Volterra integral equations, Adapted M-solutions, Dynamic coherent risk measure, Stochastic Lipschitz coefficients

1 Introduction

The literature on both static and dynamic risk measures, has been well developed since Artzner et al [2] firstly introduced the concept of coherent risk measures, see [3], [5] for more other detailed accounts. Recently, a class of static and dynamic risk measures were induced via g-expectation and conditional g-expectation respectively in [10]. g-expectation was introduced by Peng [8] as particular nonlinear expectations based on backward stochastic differential equations (BSDEs for short), which were firstly studied by Pardoux and Peng [9]. One nature characteristic of the above risk measures is time-consistency (or semi-group property), however, time-inconsistency preference usually exists in real world, see [4], [6], [12]. As to the this case, Yong [15] firstly obtained a class of continuous-time dynamic risk measures, allowing possible time-inconsistent preference, by means of backward stochastic Volterra integral equations (BSVIEs for short) in [15].

One-dimensional BSVIEs are equations of the following type defined on $[0, T]$,

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \qquad (1)$$

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where $(W_t)_{t\in[0,T]}$ is a d-dimensional Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $(\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration of (W_t), such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. The function $g : \Omega \times \Delta^c \times R \times R^d \times R^d \to R$ is generally called a generator of (1), here T is the terminal time, and the R-valued $\mathcal{F}_T$-adapted process $\psi(\cdot)$ is a terminal condition; (g, T, ψ) are the parameters of (1). A solution is a couple of processes $(Y(\cdot), Z(\cdot,\cdot))$ which have some integrability properties, depending on the framework imposed by the type of assumptions on g. Readers interested in an in-depth analysis of BSVIEs can see [14], [16], [15], [7], [13], [1] and [11], among others.

One of the assumptions in Yong [15] is that $r(\cdot)$ (the interest rate) is deterministic, otherwise, it is contradicted with the definition of translation invariance. As well known to us, in some circumstances, it is necessary that the interest rate is random, hence, in the current paper, we are dedicated to study the case of the random case by giving a general version of definition aforementioned. After that we will show a class of dynamic coherent risk measure by means of BSVIEs, allowing the interest rate to be random. Before doing this, we should prove the unique solvability of M-solution, introduced by Yong [16], of BSVIEs under stochastic Lipschitz condition,

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(s,t))ds - \int_t^T Z(t,s)dW(s), \tag{2}$$

which generalize the result in [15]. In addition, we claim that this proof is much briefer than the one in [15].

The paper is organized as follows. Section 2 is devoted to notation. In Section 3, we state a result of existence and uniqueness for BSVIEs with generators satisfying a stochastic Lipschitz condition. In Section 4, we apply the previous result to study some problems in dynamic risk measure.

2 Notation

In this paper, we define several classes of stochastic processes which we use in the sequel. We denote $\Delta^c = \{(t,s), 0 \le t \le s \le T,\}$, and $\Delta = \{(t,s), 0 \le s < t \le T,\}$. Let $L^2_{\mathcal{F}_T}[0,T]$ be the set of $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-measurable processes $\psi : [0,T] \times \Omega \to R$ such that $E\int_0^T |\psi(t)|^2 dt < \infty$. We also denote

$$\mathcal{H}^2[0,T] = L^2_{\mathbb{F}}[0,T] \times L^2(0,T; L^2_{\mathbb{F}}[0,T]),$$

where $L^2_{\mathbb{F}}[0,T]$ is the set of all adapted processes $Y : [0,T] \times \Omega \to R$ such that $E\int_0^T |Y(t)|^2 dt < \infty$, and $L^2(0,T; L^2_{\mathbb{F}}[0,T])$ is the set of all processes $Z : [0,T]^2 \times \Omega \to R$ such that for almost all $t \in [0,T]$, $z(t,\cdot) \in L^2_{\mathbb{F}}[0,T]$ satisfying

$$E\int_0^T \int_0^T |z(t,s)|^2 dsdt < \infty.$$

Now we cite some definitions introduced in [15] and [16].

Definition 2.1 *A mapping $\rho : L^2_{\mathcal{F}_T}[0,T] \to L^2_{\mathbb{F}}[0,T]$ is called a dynamic risk measure if the following hold:*

1) (Past independence) For any $\psi(\cdot)$, $\overline{\psi}(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$, if $\psi(s) = \overline{\psi}(s)$, a.s. $\omega \in \Omega$, $s \in [t,T]$, for some $t \in [0,T)$, then $\rho(t;\psi(\cdot)) = \rho(t;\overline{\psi}(\cdot))$, a.s. $\omega \in \Omega$.

2) (Monotonicity) For any $\psi(\cdot)$, $\overline{\psi}(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$, if $\psi(s) \leq \overline{\psi}(s)$, a.s. $\omega \in \Omega$, $s \in [t,T]$, for some $t \in [0,T)$, then $\rho(s;\psi(\cdot)) \geq \rho(s;\overline{\psi}(\cdot))$, a.s. $\omega \in \Omega, s \in [t,T]$.

Definition 2.2 *A dynamic risk measure $\rho : L^2_{\mathcal{F}_T}[0,T] \to L^2_{\mathbb{F}}[0,T]$ is called a coherent risk measure if the following hold: 1) (Translation invariance) There exists a deterministic integrable function $r(\cdot)$ such that for any $\psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$,*

$$\rho(t;\psi(\cdot)+c) = \rho(t;\psi(\cdot)) - ce^{\int_t^T r(s)ds}, \quad \omega \in \Omega, t \in [0,T].$$

2) (Positive homogeneity) For $\psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$ and $\lambda > 0$,

$$\rho(t;\lambda\psi(\cdot)) = \lambda\rho(t;\psi(\cdot)), \quad a.s. \quad \omega \in \Omega, \quad t \in [0,T].$$

3) (Subadditivity) For any $\psi(\cdot)$, $\overline{\psi}(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$,

$$\rho(t;\psi(\cdot)+\overline{\psi}(\cdot)) \leq \rho(t;\psi(\cdot)) + \rho(t;\overline{\psi}(\cdot)), \quad \omega \in \Omega, t \in [0,T].$$

Definition 2.3 *Let $S \in [0,T]$. A pair of $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{H}^2[S,T]$ is called an adapted M-solution of BSVIE (1) on $[S,T]$ if (1) holds in the usual Itô's sense for almost all $t \in [S,T]$ and, in addition, the following holds:*

$$Y(t) = E^{\mathcal{F}_S} Y(t) + \int_S^t Z(t,s)dW(s).$$

3 The existence and uniqueness with stochastic Lipschitz coefficient

A class of dynamic risk measures, allowing time-inconsistency preference, were induced via BSVIEs of the form, $t \in [0,T]$,

$$Y(t) = \psi(t) + \int_t^T g(t,s,Y(s),Z(s,t))ds - \int_t^T Z(t,s)dW(s). \tag{3}$$

In this section we will study the unique solvability of M-solution for (3) under more weaker assumptions, i.e., allowing the coefficients be stochastic. In addition, the proof also seems briefer than the one in [15]. So we introduce the standard assumptions as follows,

(H1) Let $g : \Delta^c \times R \times R^d \times \Omega \to R$ be $\mathcal{B}(\Delta^c \times R \times R^d) \otimes \mathcal{F}_T$-measurable such that $s \to g(t,s,y,z)$ is $\mathbb{F}$-progressively measurable for all $(t,y,z) \in [0,T] \times R \times R^d$ and

$$E\int_0^T \left(\int_t^T |g_0(t,s)|ds\right)^2 dt < \infty, \tag{4}$$

where we denote $g_0(t,s) \equiv g(t,s,0,0)$. Moreover,

$$\begin{aligned} |g(t,s,y,z) - g(t,s,\overline{y},\overline{z})| &\leq L_1(t,s)|y-\overline{y}| + L_2(t,s)|z-\overline{z}|, \\ \forall y, \overline{y} &\in R^m, \quad z, \overline{z} \in R^{m\times d}, \end{aligned} \tag{5}$$

where $L_1(t,s)$ and $L_2(t,s)$ are two non-negative $\mathcal{B}(\Delta^c)\times\mathcal{F}_T$-measurable processes such that for any

$$\int_t^T L_1^2(t,s)ds < M, \quad \left(\int_t^T L_2^q(t,s)ds\right)^{\frac{2}{q}} < M, \quad t \in [0,T],$$

for some constant M and $\frac{1}{p}+\frac{1}{q}=1$, $1<p<2$. So we obtain the following theorem,

Theorem 3.1 *Let (H1) hold, $\psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$, then (3) admits a unique M-solution in $\mathcal{H}^2[0,T]$.*

Proof. Let $\mathcal{M}^2[0,T]$ be set of elements in $\mathcal{H}^2[0,T]$ satisfying: $\forall S \in [0,T]$

$$Y(t) = E^{\mathcal{F}_S}Y(t) + \int_S^t Z(t,s)dW(s).$$

Obviously it is a closed subset of $\mathcal{H}^2[0,T]$ (see [16]). Due to the following inequality,

$$E\int_0^T e^{\beta t}dt\int_0^t |z(t,s)|^2 ds \leq E\int_0^T e^{\beta t}|y(t)|^2 dt,$$

where β is a constant, $(y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^2[0,T]$, we can introduce a new equivalent norm in $\mathcal{M}^2[0,T]$ as follows,

$$\|(y(\cdot), z(\cdot,\cdot))\|_{\mathcal{M}^2[0,T]} \equiv E\left\{\int_0^T e^{\beta t}|y(t)|^2dt + \int_0^T e^{\beta t}\int_t^T |z(t,s)|^2 dsdt\right\}^{\frac{1}{2}}.$$

Let us consider,

$$Y(t) = \psi(t) + \int_t^T g(t,s,y(s),z(s,t))ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T], \tag{6}$$

for any $\psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$ and $(y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^2[0,T]$. Hence BSVIE (6) admits a unique M-solution $(Y(\cdot), Z(\cdot,\cdot)) \in \mathcal{M}^2[0,T]$, see [16], and we can define a map $\Theta : \mathcal{M}^2[0,T] \to \mathcal{M}^2[0,T]$ by

$$\Theta(y(\cdot), z(\cdot,\cdot)) = (Y(\cdot), Z(\cdot,\cdot)), \quad \forall (y(\cdot), z(\cdot,\cdot)) \in \mathcal{M}^2[0,T].$$

Let $(\overline{y}(\cdot),\overline{z}(\cdot,\cdot)) \in \mathcal{M}^2[0,T]$ and $\Theta(\overline{y}(\cdot),\overline{z}(\cdot,\cdot)) = (\overline{Y}(\cdot),\overline{Z}(\cdot,\cdot))$. Obviously we obtain that

$$
\begin{aligned}
& E\int_0^T e^{\beta t}|Y(t)-\overline{Y}(t)|^2dt + E\int_0^T e^{\beta t}dt\int_t^T |Z(t,s)-\overline{Z}(t,s)|^2ds \\
\leq\ & CE\int_0^T e^{\beta t}\left\{\int_t^T |g(t,s,y(s),z(s,t))-g(t,s,\overline{y}(s),\overline{z}(s,t))|ds\right\}^2 dt \\
\leq\ & CE\int_0^T e^{\beta t}\left\{\int_t^T L_1(t,s)|y(s)-\overline{y}(s)|ds\right\}^2 dt \\
& +CE\int_0^T e^{\beta t}\left\{\int_t^T L_2(t,s)|z(s,t)-\overline{z}(s,t)|ds\right\}^2 dt \\
\leq\ & CE\int_0^T e^{\beta t}\left(\int_t^T L_1^2(t,s)ds\right)\int_t^T |y(s)-\overline{y}(s)|^2dsdt \\
& +CE\int_0^T e^{\beta t}\left(\int_t^T L_2^{q'}(t,s)ds\right)^{\frac{2}{q'}}\left(\int_t^T |z(s,t)-\overline{z}(s,t)|^{p'}ds\right)^{\frac{2}{p'}} dt \\
\leq\ & CE\int_0^T |y(s)-\overline{y}(s)|^2ds\int_0^s e^{\beta t}dt + C\left[\frac{1}{\beta}\right]^{\frac{2-p'}{p'}} E\int_0^T ds\int_t^T e^{\beta s}|z(s,t)-\overline{z}(s,t)|^2dt \\
\leq\ & \frac{C}{\beta}E\int_0^T e^{\beta s}|y(s)-\overline{y}(s)|^2ds + C\left[\frac{1}{\beta}\right]^{\frac{2-p'}{p'}} E\int_0^T e^{\beta t}dt\int_0^t |z(t,s)-\overline{z}(t,s)|^2ds \\
\leq\ & \left(\frac{C}{\beta}+C\left[\frac{1}{\beta}\right]^{\frac{2-p'}{p'}}\right)E\int_0^T e^{\beta s}|y(s)-\overline{y}(s)|^2ds,
\end{aligned}
$$

where $1<p'<2$, $\frac{1}{p'}+\frac{1}{q'}=1$. Note that in above we use the following relation, for any $1<p'<2$, and $r>0$,

$$
\begin{aligned}
& \left[\int_t^T |z(s,t)-\overline{z}(s,t)|^{p'}ds\right]^{\frac{2}{p'}} \\
\leq\ & \left[\int_t^T e^{-rs\frac{2}{2-p'}}ds\right]^{\frac{2-p'}{p'}}\int_t^T e^{rs\frac{2}{p'}}|z(s,t)-\overline{z}(s,t)|^2ds \\
\leq\ & \left[\frac{1}{r}\right]^{\frac{2-p'}{p'}}\left[\frac{2-p'}{p}\right]^{\frac{2-p'}{p'}} e^{-rt\frac{2}{p'}}\int_t^T e^{rs\frac{2}{p'}}|z(s,t)-\overline{z}(s,t)|^2ds, \qquad (7)
\end{aligned}
$$

Then we can choose a β, so that the map Θ is a contraction, and (6) admits a unique M-solution. $\square$

4 Applications in finance

In what follows, we define

$$
\rho(t,\psi(\cdot)) = Y(t), \quad \forall t\in[0,T], \qquad (8)
$$

where $(Y(\cdot), Z(\cdot,\cdot))$ is the unique adapted M-solution of (3).

Now we cite a proposition from [15].

Proposition 4.1 *Let us consider the following form of BSVIE, $t \in [0,T]$,*

$$Y(t) = -\psi(t) + \int_t^T (r_1(s)Y(s) + g(t,s,Z(s,t)))ds - \int_t^T Z(t,s)dW(s). \qquad (9)$$

If $r_1(s)$ is a bounded and deterministic function, then $\rho(\cdot)$ defined by (8) is a dynamic coherent risk measure if $z \mapsto g(t,s,z)$ is positively homogeneous and sub-additive.

In Proposition 4.1, due to the way of defining translation invariance in definition 2.2, $r(\cdot)$ must be a deterministic function. In fact, we require $t \mapsto \rho(t;\psi(\cdot)+c)$ is $\mathbb{F}$-adapted, and if we allow $r(\cdot)$ to be an $\mathbb{F}$-adapted process, the axiom will become controversial. Hence if we would like to consider the random case, we should replace the axiom to a more general form. So we have,

1')There exists a $\mathbb{F}$-adapted process $Y_0(t)$ such that for any $\psi(\cdot) \in L^2_{\mathcal{F}_T}[0,T]$,

$$\rho(t;\psi(\cdot)+c) = \rho(t;\psi(\cdot)) - Y_0(t), \quad \omega \in \Omega, t \in [0,T].$$

Now we give a class of dynamic coherent risk measure via certain BSVIEs. We have,

Theorem 4.1 *let us consider*

$$Y(t) = -\psi(t) + \int_t^T l_2(t,s)Z(s,t) + l_1(t,s)Y(s)ds + \int_t^T Z(t,s)dW(s), \qquad (10)$$

where $l_i, (i=1,2)$ are two bounded processes such that $s \mapsto l_i(t,s), (i=1,2)$ are $\mathbb{F}$-adapted for almost every $t \in [0,T]$, then $\rho(\cdot)$ is a dynamic coherent risk measure.

Proof. The result is obvious, so we omit it. $\square$

Remark 4.1 *In Theorem 4.1, the coefficient of $Y(s)$ is random, which generalizes the situation in Proposition 4.1. However, as we will assert below, in this case g usually can not be a general form as in BSVIE (9).*

Let us consider the following two equations,

$$\begin{cases} Y^{\psi}(t) = -\psi(t) + \int_t^T (l'(t,s)Y^{\psi}(s)ds \\ \qquad\qquad + \int_t^T g(t,s,Z^{\psi}(s,t)))ds - \int_t^T Z^{\psi}(t,s)dW(s), \\ Y^{\psi+c}(t) = -\psi(t) - c + \int_t^T l'(t,s)Y^{\psi+c}(s)ds \\ \qquad\qquad + \int_t^T g(t,s,Z^{\psi+c}(s,t))ds - \int_t^T Z^{\psi+c}(t,s)dW(s), \end{cases}$$

where l' is a process such that $s \mapsto l'(t,s)$ is $\mathbb{F}$-adapted for almost every $t \in [0,T]$, $(Y^{\psi}(\cdot), Z^{\psi}(\cdot,\cdot)$ and $(Y^{\psi+c}(\cdot), Z^{\psi+c}(\cdot,\cdot)$ are the unique M-solution of the above two BSVIEs. We denote

$$Y'(t) = Y^{\psi+c}(t) - Y^{\psi}(t), Z'(t,s) = Z^{\psi+c}(t,s) - Z^{\psi}(t,s), \quad t,s \in [0,T].$$

Thus we deduce that

$$\begin{aligned} Y'(t) &= -c + \int_t^T (g(t,s,Z^{\psi+c}(s,t)) - g(t,s,Z^{\psi}(s,t)))ds \\ &\quad + \int_t^T l'(t,s)Y'(s)ds + \int_t^T Z'(t,s)dW(s), \end{aligned} \tag{11}$$

The definition of M-solution implies

$$\begin{cases} Y^{\psi}(t) = EY^{\psi}(t) + \int_0^t Z^{\psi}(t,s)dW(s), \\ Y^{\psi+c}(t) = EY^{\psi+c}(t) + \int_0^t Z^{\psi+c}(t,s)dW(s), \end{cases}$$

so we have

$$Z'(t,s) = Z^{\psi+c}(t,s) - Z^{\psi}(t,s), \quad (t,s) \in \Delta,$$

thus we can rewrite (11) as

$$\begin{aligned} Y'(t) &= -c + \int_t^T (g(t,s,Z^{\psi}(s,t) + Z'(s,t)) - g(t,s,Z^{\psi}(s,t)))ds \\ &\quad + \int_t^T l'(t,s)Y'(s)ds + \int_t^T Z'(t,s)dW(s), \end{aligned} \tag{12}$$

We assume (Y^{ψ}, Z^{ψ}) is known, then under (H1), (12) admits a unique M-solution (Y', Z'), which maybe depends on (Y^{ψ}, Z^{ψ}). Obviously $Y' = Y^{\psi+c} - Y^{\psi}$, which means means that when the total wealth ψ is known to be increased by an amount $c > 0$ (if $c < 0$, it means a decrease), then the dynamic risk will be decreased (or increase) Y' which maybe depends on the total wealth ψ. Next we give two special cases.

(1) When $l'(t,s)$ is independent of ω, let us consider the following backward Volterra integral equation (BVIE for short),

$$Y^*(t) = -c + \int_t^T l'(t,s)Y^*(s)ds, \quad t \in [0,T]. \tag{13}$$

Obviously by the fixed point theorem as above, (13) admits a unique deterministic solution when $l'(t,s)$ satisfies the assumption in (H1). It is easy to check that $(Y^*, 0)$ is the unique M-solution of (13) when l' is a deterministic function. In fact, if Y^* is deterministic, then $Z^*(t,s) = 0, (t,s) \in \Delta$. After substituting (Y^*, Z^*) into BSVIE (12), we obtain

$$Z^{\psi+c}(t,s) = Z^{\psi}(t,s), g(t,s,Z^{\psi+c}(s,t) = g(t,s,Z^{\psi}(s,t), \quad (t,s) \in \Delta,$$

thus $(Y^*,0)$ is a adapted solution of (12), moreover, it is the unique M-solution. So in this case, g can be a general form $g(t,s,z)$. When l' is independent of t, then $Y^*(t) = -ce^{\int_t^T r(u)du}$, and we can get the result in [15].

(2) When l' depends on ω, if the value of (Y', Z') is independent of (Y^ψ, Z^ψ), g usually can not be the general form $g(t,s,z)$ as above. On the one hand, the similar result as case (1) above no longer holds. In fact, in this case, Y' usually depends on ω, i.e., there exists a $A \subseteq [0,T]$ satisfying $\lambda(A) > 0$, such that for $t \in A$, $P\{\omega \in \Omega, Y'(t) \neq EY'(t)\} > 0$, where λ is the Lebeague measure, then

$$E\int_0^T \int_0^t |Z'(t,s)|^2 ds dt = E\int_0^T |Y'(t) - EY'(t)|^2 dt > 0,$$

which means that there must exist a set $B \subseteq \Delta$, such that $\forall (t,s) \in B$,

$$P(\{Z'(t,s) \neq 0\}) > 0, \quad \lambda(B) > 0.$$

Then for a general form of $g(t,s,z)$,

$$g(t,s,Z'(s,t) + Z^\psi(s,t)) \neq g(t,s,Z^\psi(s,t)),$$

which means Y' maybe depends on ψ. For example, if we let $l(s) = \sin W(s)$, $c \neq 0$, and let's consider the equation below,

$$Y^c(t) = -c + \int_t^T \sin W(s) Y^c(s) ds + \int_t^T Z^c(t,s) dW(s). \tag{14}$$

By theorem 3.1 it admits a unique M-solution. If $Y^c = EY^c$, a.e., a.s., then

$$Y^c(t) \quad = \quad -c + \int_t^T E \sin W(s) Y^c(s) ds \tag{15}$$

Since $E\sin W(t) = 0$ implies $Y^c(t) = -c$, then $Z(t,s) = 0$, $(t,s) \in [0,T]^2$, thus we have for almost any $t \in [0,T]$, $\int_t^T \sin W(s) ds = 0$, which means that for almost any $t \in [0,T]$, $\sin W(t) = 0$, obviously it is a contradiction. On the other hand, if $g(t,s,Z'(s,t) + Z^\psi(s,t)) - g(t,s,Z^\psi(s,t))$ is independent of $Z^\psi(s,t)$, roughly speaking, the only good case for this is g is a linear function of z, otherwise the result will not hold. For example, if we let $g(t,s,z) = l(t,s)z^2$, which is a convex function for z, then

$$g(t,s,Z'(s,t) + Z^\psi(s,t)) - g(t,s,Z^\psi(s,t)) = 2l(t,s)Z'(s,t)Z^\psi(s,t) + Z'^2(s,t).$$

Obviously the value of Y_0 depends on ψ.

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